On the Laplace transform of perpetuities with thin tails

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Abstract

We consider the random variables R which are solutions of the distributional equation $R \overset{\mathcal{L}}{=} MR+Q$, where (Q, M) is independent of R and $|M| \leqslant 1$. Goldie and Grübel showed that the tails of R are no heavier than exponential. Alsmeyer and *al* provide a complete description of the domain of the Laplace transform of R. We present here a simple proof in a particular case and an extension to the Markovian case.

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1 Introduction

We define on some probability space $(\Omega, \mathcal{A}, \mathbb{P})$ a couple of random variables (M, Q), a sequence $(M_n, Q_n)_{n\geqslant 0}$ of independent and identically distributed random vectors with the same law as (M, Q), and R_0 a random variable independent of the sequence $(M_n, Q_n)_{n\geqslant 0}$. Define the sequence $(R_n)_{n\geqslant 0}$ by

$$R_{n+1} = M_n R_n + Q_n, \tag{1}$$

for any $n \geqslant 0$. This sequence has been extensively studied in the last decades. Under weak assumptions (see [8]) which are obviously fullfilled in our setting, it can be shown that the sequence $(R_n)_{n\geqslant 0}$ converges almost surely to a random variable R such that

$$R \overset{\mathcal{L}}{=} MR + Q, \tag{2}$$

where R is independent of (M, Q).

In [7], Kesten established that R is in general heavy-tailed (*i.e.* not all the moments of R are finite) even if Q is light-tailed as soon as $|M|$ can be greater than 1. Nevertheless, Goldie and Grübel [4] have shown that R can have some exponential moments if $|M| \leqslant 1$. In particular, if Q and M are nonnegative the following result holds.

Theorem 1.1 (Goldie, Grübel [4]). *Assume that*

$$\mathbb{P}(Q \geqslant 0, 0 \leqslant M \leqslant 1) = 1, \quad \mathbb{P}(M < 1) > 0$$

and that there is $v_Q > 0$ (possibly infinite) such that

$$\mathbb{E}\big(e^{vQ}\big) \begin{cases} < +\infty & \text{if } v < v_Q, \\ = +\infty & \text{if } v > v_Q. \end{cases} \tag{3}$$

Then, the Laplace transform $v \mapsto \mathbb{E}\big(e^{vR}\big)$ of the solution R of (2) is finite on the set $(-\infty, v_{GG})$ with $v_{GG} = v_Q \wedge \sup\big\{v \geqslant 0,\ \mathbb{E}(e^{vQ}M) < 1\big\}$.

In fact, the domain of the Laplace transform of R is larger than $(-\infty, v_{GG})$. In [1], a full description of this domain is established. Let us provide a simple proof under the assumptions of Theorem 1.1.

2 The main result

Theorem 2.1. *Under the assumptions of Theorem 1.1, assuming furthermore that R_0 is non-negative and has all its exponential moments finite, then*

$$\sup_{n\geqslant 0} \mathbb{E}\big(e^{vR_n}\big) < +\infty \quad \textit{and} \quad \mathbb{E}\big(e^{vR}\big) < +\infty$$

for any $v < v_c$, where

$$v_c = v_Q \wedge \sup\{v \geqslant 0,\, \mathbb{E}\big(e^{vQ}1_{\{M=1\}}\big) < 1\}.$$

Moreover, for any $v > v_c$, $\sup_{n\geqslant 0} \mathbb{E}\big(e^{vR_n}\big) = +\infty$ and $\mathbb{E}\big(e^{vR}\big) = +\infty$.

For other recent generalizations of [4], the interested reader is referred to [6], where the authors give sharper results than ours on the tails for some specific examples.

Proof of Theorem 2.1. Let us start this section with the main lines of the proof of Theorem 1.1 of Goldie and Grübel [4]. For $\rho > 0$, let $\mathcal{M}_\rho$ be the set of probability measures on $\mathbb{R}_+$ with finite exponential moment of order ρ, and d_ρ a distance defined on $\mathcal{M}_\rho$ by: for $\mu, \nu \in \mathcal{M}_\rho$,

$$d_\rho(\mu, \nu) = \int_0^\infty e^{\rho u} |\mu[u, \infty) - \nu[u, \infty)|\, du.$$

Define the application T on $\mathcal{M}_\rho$ as follows: for X with law $\mu \in \mathcal{M}_\rho$, $T\mu$ is the law of $Q + MX$ with (M, Q) independent of X. It is shown in [4] that,

$$d_\rho(T\mu, T\nu) \leqslant \mathbb{E}(e^{\rho Q} M) d_\rho(\mu, \nu).$$

Since

$$\mathbb{E}\big(e^{vX}\big) = v \int_0^\infty e^{vu} \mathbb{P}(X \geqslant u)\, du,$$

one can show that, for any $n \geqslant 0$ and $v < \min(v_0, v_Q)$ with $v_0 = \sup\{v \geqslant 0,\, \mathbb{E}\big(e^{vQ}M\big) < 1\}$,

$$\mathbb{E}\big(e^{vR_n}\big) \leqslant v \frac{1 - \mathbb{E}(e^{vQ}M)^n}{1 - \mathbb{E}(e^{vQ}M)} d_v(T\mu_0, \mu_0) + \mathbb{E}\big(e^{vR_0}\big).$$

In others words, Goldie and Grübel [4] established that for any $v < \min(v_0, v_Q)$, $(\mathbb{E}(e^{vR_n}))_n$ is uniformly bounded. This estimate can be extended to a larger domain.

Let us define $v_1 = \sup\{v \geqslant 0,\, \mathbb{E}\big(e^{vQ}1_{\{M=1\}}\big) < 1\}$ and $v_c = \min(v_1, v_Q)$. Let us fix $v < v_c$ and choose $\varepsilon > 0$ such that

$$\rho := \mathbb{E}(e^{vQ}1_{\{1-\varepsilon < M \leqslant 1\}}) < 1.$$

Then we get, for any $n \geqslant 0$,

$$\begin{aligned} L_{n+1}(v) := \mathbb{E}(e^{vR_{n+1}}) &= \mathbb{E}(e^{v(M_nR_n+Q_n)}) \\ &\leqslant \mathbb{E}(e^{v((1-\varepsilon)R_n+Q_n)}1_{\{M_n\leqslant 1-\varepsilon\}}) + \mathbb{E}(e^{v(R_n+Q_n)}1_{\{1-\varepsilon<M_n\leqslant 1\}}) \\ &\leqslant L_n((1-\varepsilon)v)L_Q(v) + \rho L_n(v) \end{aligned}$$

where $L_Q(v) = \mathbb{E}(e^{vQ})$. By iteration of this estimate, one gets for any $n \geqslant 0$

$$L_n(v) \leqslant \Big(\sum_{k=0}^{n-1} \rho^k L_{n-k}((1-\varepsilon)v)\Big) L_Q(v) + \rho^n L_0(v)\,.$$

Let us notice that we have in fact more: for the same ε, and for any $\tilde{v} \leqslant v$, $\tilde{\rho} := \mathbb{E}(e^{\tilde{v}Q}1_{\{1-\varepsilon<M\leqslant 1\}}) < \rho$, hence, by the same method as before,

$$L_n(\tilde{v}) \leqslant \Big(\sum_{k=0}^{n-1} \rho^k L_{n-k}((1-\varepsilon)\tilde{v})\Big) L_Q(\tilde{v}) + \rho^n L_0(\tilde{v})\,. \tag{4}$$

Let us define $\overline{L} = \sup_{n\geqslant 0} L_n$. Taking the supremum over n in (4), one gets for any $\tilde{v} \leqslant v$

$$\overline{L}(\tilde{v}) \leqslant \frac{1}{1-\rho}\overline{L}((1-\varepsilon)\tilde{v})L_Q(\tilde{v}) + L_0(\tilde{v})\,. \tag{5}$$

There is $k \in \mathbb{N}$ such that $(1-\varepsilon)^k v < v_0$, hence $\overline{L}((1-\varepsilon)^k v) < +\infty$. Applying k times estimate (5), one then obtains immediatly that $\overline{L}(v) < +\infty$, which achieves the first part of the proof.

On the other hand, if $v > v_Q$, $R_1 \geqslant Q_0$ immediatly implies $L_1(v) = +\infty$; if $v > v_1$, $\rho_0 := \mathbb{E}(e^{vQ}1_{\{M=1\}}) > 1$ (except in a trivial case, left to the reader) and, for all $n \geqslant 0$,

$$L_{n+1}(v) \geqslant \rho_0 L_n(v)\,,$$

implying that $\overline{L}(v) = +\infty$. □

3 Some extensions and perspectives

What happens if the random variables $(M_n, Q_n)_{n\geqslant 0}$ are no longer independent? We provide here a partial result under a Markovian assumption when the contractive term M is less than 1.

Let us introduce $X = (X_n)_{n\geqslant 0}$ an irreducible recurrent Markov process with finite space E and $((M_n(x), Q_n(x))_{x\in E})_{n\geqslant 0}$ a sequence of i.i.d. random vectors supposed to be independent of X. We assume that, for all $x \in E$,

$$\mathbb{P}(0 \leqslant M(x) < 1) = 1,$$

but we do not assume in the sequel that Q is non negative. The sequence $(R_n)_{n\geqslant 0}$ is defined by

$$R_{n+1} = M_n(X_n)R_n + Q_n(X_n),$$

R_0 being arbitrary (with all exponential moments). Notice that the process $(X_n, R_n)_{n\geqslant 0}$ is a Markov process whereas $(R_n)_{n\geqslant 0}$ is not (in general).

Proposition 3.1. *Introduce* $\underline{v} = \inf_{x\in E} v_{|Q(x)|}$*, with* $v_{|Q(x)|}$ *defined as in (3). For any* $v < \underline{v}$*,*

$$\sup_{n\geqslant 0} \mathbb{E}\left(e^{v|R_n|}\right) < +\infty.$$

Moreover, if $v > \underline{v}$*, then this supremum is infinite.*

Proof. Let us introduce $\overline{M}_n = \max_{x\in E} M_n(x)$ and $\overline{Q}_n = \max_{x\in E} |Q_n(x)|$. The random variables $((\overline{M}_n, \overline{Q}_n))_{n\geqslant 0}$ are i.i.d. Define the sequence $(\overline{R}_n)_{n\geqslant 0}$ by

$$\overline{R}_0 = |R_0| \quad \text{and} \quad \overline{R}_{n+1} = \overline{M}_n \overline{R}_n + \overline{Q}_n \quad \text{for } n \geqslant 1.$$

Obviously, $|R_n| \leqslant \overline{R}_n$ for all $n \geqslant 0$. Thus it is sufficient to study the Laplace transforms of $(\overline{R}_n)_{n\geqslant 0}$. On the other hand, Theorem 2.1 ensures that $\left(\mathbb{E}\left(e^{v\overline{R}_n}\right)\right)_n$ is uniformly bounded as soon as $v < \overline{v}_c = \min(\overline{v}_1, v_{\overline{Q}})$ with $\overline{v}_1 = \sup\{v \geqslant 0 \,:\, \mathbb{E}(e^{v\overline{Q}} 1_{\{\overline{M}=1\}}) < 1\} > 0$. In our case, $\overline{v}_1$ is infinite since $\mathbb{P}(\overline{M} < 1) = 1$. At last, for $v \geqslant 0$,

$$\sup_{x\in E} \mathbb{E}\left(e^{v|Q(x)|}\right) \leqslant \mathbb{E}\left(e^{v\overline{Q}}\right) = \mathbb{E}\left(\sup_{x\in E} e^{v|Q(x)|}\right) \leqslant \sum_{x\in E} \mathbb{E}\left(e^{v|Q(x)|}\right).$$

Thus $v_{\overline{Q}} = \inf_{x\in E} v_{|Q(x)|}$.

On the other hand, choose $v > \underline{v}$. There exists $x_0 \in E$ such that $\mathbb{E}\big(e^{v|Q(x_0)|}\big)$ is infinite. Then, for any $n \geqslant 0$,

$$\begin{aligned}
\mathbb{E}\left(e^{v|R_{n+1}|}\right) &\geqslant \mathbb{E}\left(e^{v|R_{n+1}|} \mathbb{1}_{\{X_n=x_0\}}\right) \\
&\geqslant \mathbb{E}\left(e^{-v|R_n|} e^{v|Q_n(x_0)|} \mathbb{1}_{\{X_n=x_0\}}\right) \\
&\geqslant \mathbb{E}\left(\mathbb{1}_{\{X_n=x_0\}} e^{-v|R_n|}\right) \mathbb{E}\left(e^{v|Q_n(x_0)|}\right).
\end{aligned}$$

The recurrence of X ensures that $\big\{n \geqslant 0,\ \mathbb{E}\big(e^{v|R_n|}\big) = +\infty\big\}$ is infinite. $\square$

Remark 3.2. *In [2], we use the previous estimates to improve the results of [5, 3] on the tails of the invariant measure of a diffusion process with Markov switching.*

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